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Thermodynamics of the quantum Perk–Schultz model*

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Abstract. The quantum mechanical analogue of the classical Perk–Schultz model is considered which comprises the Uimin–Sutherland model and the integrable t–J chain. The quantum transfer matrix of these systems is established and the eigenvalue equations are obtained by an algebraic Bethe ansatz. Only the largest eigenvalue is needed for the calculation of the free energy of the quantum chain at finite temperature. The Bethe ansatz equations for the leading eigenvalue are transformed into a set of integral equations for some appropriately defined auxiliary functions. Furthermore, the eigenvalue of the quantum transfer matrix is expressed in terms of these functions. The integral formulation allows for taking the limit of infinite Trotter–Suzuki number analytically. The low-temperature limit of the free energy is obtained analytically and for intermediate temperatures numerical results are presented.

1. Introduction

In [1] a new method for the treatment of thermodynamics of integrable one-dimensional quantum-mechanical models has been introduced. In contrast to the traditional method [2], which is based on a direct evaluation of the partition function by taking into account all energy levels of the system, the new approach makes use of the well known relationship between one-dimensional quantum mechanical and two-dimensional classical systems. Via a Trotter–Suzuki mapping the finite-temperature partition function of the quantum spin chain is mapped onto the partition function of a classical inhomogeneous system, which can be expressed in terms of the eigenvalues of a so-called quantum transfer matrix. The free energy of the quantum chain for arbitrary temperatures can be related to the largest eigenvalue of this matrix, while correlation lengths of the system can be expressed in terms of the next-largest eigenvalues. This concept has been applied successfully to spin-$\frac{1}{2}$ chains as well as to the Hubbard model [3–5].

In this paper we study the quantum mechanical analogue of the classical Perk–Schultz model, which was introduced as an integrable multi-component generalization of the six-vertex model [6]. The transfer matrix of this model was diagonalized in [7] and the corresponding Hamiltonian limit was studied in [8] leading to a whole family of integrable one-dimensional systems. It turns out that several interesting fermionic models are contained in this family, among them an anisotropic generalization of the supersymmetric t–J model [8, 9] and the Essler–Korepin–Schoutens model [10]. Here we generalize the algebraic Bethe ansatz method from [7] to an inhomogeneous system and thus find an equation for the eigenvalues of the quantum transfer matrix. For the particularly interesting case of a three-component system we transform the equations for the largest eigenvalue into a system

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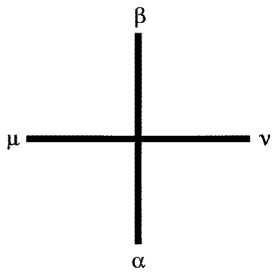

Figure 1. Variables around a vertex.

of nonlinear integral equations, from which the low-temperature behaviour of the free energy of the corresponding quantum chains can be calculated. We also present numerical results for the free energy at arbitrary temperatures of one particular quantum chain.

2. The classical model and its Hamiltonian limit

We consider a square lattice with periodic boundary conditions. Each bond of the lattice is occupied by a variable taking on values $0, \ldots, q-1$. The only allowed arrow configurations are such that the values of the two variables on the lower and left-hand bond of a vertex α and μ are the same as those of the variables on the upper and right-hand bonds β and ν (cf figure 1). This implies that only $2q^2 - q$ of the q^4 Boltzmann weights $R^{\mu\nu}_{\alpha\beta}$ are nonzero and these are conveniently parametrized as

$$\begin{aligned} R^{\alpha\alpha}_{\alpha\alpha}(v) &= \sinh(\eta + \epsilon_\alpha v)/\sinh\eta \\ R^{\mu\mu}_{\alpha\alpha}(v) &= \epsilon_{\alpha\mu}\sinh v/\sinh\eta \\ R^{\mu\alpha}_{\alpha\mu}(v) &= \exp(\mathrm{sign}(\alpha-\mu)v) \end{aligned} \tag{1}$$

with

$$\epsilon_\alpha = \pm 1 \qquad \epsilon_{\alpha\mu} = \epsilon_{\mu\alpha} = \epsilon_\alpha\epsilon_\mu \qquad \alpha, \mu = 0, \ldots, q-1. \tag{2}$$

The weights (1) satisfy the Yang–Baxter equations [11] which implies that the row-to-row transfer matrices

$$T^{\beta}_{\alpha}(v) = \sum_{\boldsymbol{\mu}} \prod_{i=1}^{N} R^{\mu_i\mu_{i+1}}_{\alpha_i\beta_i}(v) \qquad \boldsymbol{\alpha} = \{\alpha_1, \ldots \alpha_N\}, \ldots \tag{3}$$

form a commuting family. At the point $v = 0$, where $T(v)$ reduces to the right-shift operator, the Hamiltonian limit of the classical model can be obtained by taking the logarithmic derivative of the transfer matrix

$$H = \frac{\mathrm{d}}{\mathrm{d}v}\{\ln[T(v)]\}|_{v=0} = \sum_{i=1}^{N} h_i \tag{4}$$

where the matrix elements of the local Hamiltonian h_i are given by

$$(h_i)^{\alpha\alpha}_{\alpha\alpha} = \epsilon_\alpha \coth\eta \qquad (h_i)^{\alpha\mu}_{\mu\alpha} = \epsilon_{\alpha\mu}/\sinh\eta \qquad (h_i)^{\mu\alpha}_{\mu\alpha} = \mathrm{sign}(\alpha-\mu). \tag{5}$$

The Hamiltonian (5) can be interpreted as describing a system of particles with f fermionic and $b = q - f$ bosonic degrees of freedom. For $f = q - 1$ one recovers the 'generalized t–J models' of [8]. In particular for $q = 3$ and $\epsilon_1 = \epsilon_2 = 1, \epsilon_3 = -1$ one finds in the isotropic limit ($\eta \to 0$) the supersymmetric t–J model [12], for general η the integrable t–J

chain with anisotropic interactions [9], for $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$ the Uimin–Sutherland model [13, 14], and for $q = 4$ and $\epsilon_1 = \epsilon_4 = 1, \epsilon_2 = \epsilon_3 = -1$ the Essler–Korepin–Schoutens model [10].

3. Derivation of the quantum transfer matrix

As mentioned before the essential ingredient of our treatment of the thermodynamics of the Hamiltonian (5) is a mapping of the partition function at finite temperature onto the partition function of a classical model on an inhomogeneous lattice. In contrast to the solid-on-solid and eight-vertex models, which were treated in [3] and [4], there exists only one value of the spectral parameter at which the Hamiltonian limit of the transfer matrix (1) can be taken. It is therefore necessary to consider a second family of transfer matrices $\overline{T}(v)$ obtained from (3) by substituting the rotated weights $\overline{R}(v)$ given by

$$\overline{R}^{\mu\nu}_{\alpha\beta}(v) = R^{\alpha\beta}_{\nu\mu}(v) \tag{6}$$

for $R(v)$. Obviously $\overline{T}(0)$ is the left-shift operator and the Hamiltonian obtained by taking the logarithmic derivative of $\overline{T}(v)$ at $v = 0$ is again H. To summarize we have

$$\begin{aligned} T(-v) &= T_R \mathrm{e}^{-vH+\mathcal{O}(v^2)} \\ \overline{T}(-v) &= T_L \mathrm{e}^{-vH+\mathcal{O}(v^2)} \end{aligned} \tag{7}$$

and consequently

$$[T(-\beta/N)\overline{T}(-\beta/N)]^{N/2} = \mathrm{e}^{-\beta H+\mathcal{O}(1/N)}. \tag{8}$$

The partition function of the Hamiltonian can now be written as

$$Z = \lim_{N\to\infty} \mathrm{Tr}[T(-\beta/N)\overline{T}(-\beta/N)]^{N/2}. \tag{9}$$

Thus, the partition function of the quantum chain at finite temperature is given by the partition function of an inhomogeneous Perk–Schultz model with alternating rows. For the calculation of this partition function the usual row-to-row transfer matrix (given by the square bracket in (9)) is not very useful as it obviously does not possess a gap between the largest and the next-largest eigenvalue in the limit $N \to \infty$. From this point of view the column-to-column transfer matrix (quantum transfer matrix) is best adapted. For this and the interchangeability of the limits $N \to \infty$ and chain length $L \to \infty$ the reader is referred to [1, 15]. The quantum transfer matrix may be rewritten as a row-to-row transfer matrix of the form

$$T^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\mu}} \prod_{i=1}^{N/2} R^{\mu_{2i-1}\mu_{2i}}_{\alpha_{2i-1}\beta_{2i-1}}(u)\tilde{R}^{\mu_{2i}\mu_{2i+1}}_{\alpha_{2i}\beta_{2i}}(-u) \tag{10}$$

with $u = -\beta/N$, and

$$\tilde{R}^{\mu\nu}_{\alpha\beta}(v) = R^{\beta\alpha}_{\mu\nu}(-v) \tag{11}$$

where the additional minus sign has been introduced for later convenience. Due to the fact that $R(v)$ and $\tilde{R}(v)$ satisfy the Yang–Baxter equations

$$R^{\lambda b}_{\mu c}(v-\bar{v})X^{b\lambda'}_{\alpha a}(v)X^{c\mu'}_{a\alpha'}(\bar{v}) = X^{\mu c}_{\alpha a}(\bar{v})X^{\lambda b}_{a\alpha'}(v)R^{b\lambda'}_{c\mu'}(v-\bar{v}) \tag{12}$$

with $X = R$ or $\tilde{R}$, and the *same* intertwiner $R(v-\bar{v})$, the quantum transfer matrix T can be embedded into a commuting family, namely

$$T(v) = \sum_{\boldsymbol{\mu}} \prod_{i=1}^{N/2} R^{\mu_{2i-1}\mu_{2i}}_{\alpha_{2i-1}\beta_{2i-1}}(v+\beta/N)\tilde{R}^{\mu_{2i}\mu_{2i+1}}_{\alpha_{2i}\beta_{2i}}(v-\beta/N) \tag{13}$$

(i.e. $T(0) = T$), so the integrability of the model is preserved. The free energy per site is $f = -1/\beta \lim_{L\to\infty} \ln Z/L$, where first the limit $N \to \infty$ has to be taken and then $L \to \infty$. We are allowed to interchange these limits due to the theorems in [15]. Standard reasoning then yields

$$f = -\frac{1}{\beta} \lim_{N\to\infty} \ln \Lambda_{\max} \tag{14}$$

where $\Lambda_{\max}$ denotes the largest eigenvalue of $T(0)$. The next-leading eigenvalues give the correlation lengths at finite temperature

$$\frac{1}{\xi} = -\lim_{N\to\infty} \ln \left| \frac{\Lambda}{\Lambda_{\max}} \right|. \tag{15}$$

However, the evaluation of the correlation lengths will be the subject of a future paper as the additional technical means would go far beyond the scope of the present work.

4. Eigenvalues of the quantum transfer matrix

Our aim in this section is to diagonalize the inhomogeneous transfer matrix (13). The homogeneous variant of this problem was solved in [7] using an algebraic Bethe ansatz. Here we will show how in the inhomogeneous case elementary (i.e. one-particle) excitations above a reference state can be constructed by this method. The results found in this way can be generalized to the general (n-particle) case.

We define the elements of the monodromy matrix $\mathcal{L}(v)$ by an alternating product of R and $\tilde{R}$:

$$\mathcal{L}^{\beta\lambda'}_{\alpha\lambda}(v) = R^{\lambda\mu_2}_{\alpha_1\beta_1}(v+u)\tilde{R}^{\mu_2\mu_3}_{\alpha_2\beta_2}(v-u)\ldots R^{\mu_{N-1}\mu_N}_{\alpha_{N-1}\beta_{N-1}}(v+u)\tilde{R}^{\mu_N\lambda'}_{\alpha_N\beta_N}(v-u) \tag{16}$$

such that

$$T^{\beta}_{\alpha}(v) = \mathrm{Tr}_{\lambda}[\mathcal{L}^{\beta}_{\alpha}(v)] \equiv \sum_{\lambda} \mathcal{L}^{\beta\lambda}_{\alpha\lambda}(v). \tag{17}$$

From the Yang–Baxter equations (12), one finds for the elements of the monodromy matrix

$$R^{\lambda b}_{\mu c}(v-\bar{v})\mathcal{L}^{\lambda'}_{b}(v)\mathcal{L}^{\mu'}_{c}(\bar{v}) = \mathcal{L}^{c}_{\mu}(\bar{v})\mathcal{L}^{b}_{\lambda}(v)R^{b\lambda'}_{c\mu'}(v-\bar{v}). \tag{18}$$

From now we will focus on the case $q = 3$. If we take the 'Néel state' $|\mathbf{12}\rangle = |1212\ldots12\rangle$ as a reference state, we find that the action of the monodromy matrix on this state is given by

$$\mathcal{L}(v)|\mathbf{12}\rangle = \begin{pmatrix} \alpha_1(v)|\mathbf{12}\rangle & * & * \\ 0 & \alpha_2(v)|\mathbf{12}\rangle & 0 \\ 0 & * & \alpha_3(v)|\mathbf{12}\rangle \end{pmatrix} \tag{19}$$

with

$$\begin{aligned} \alpha_1(v) &= [R^{11}_{11}(u+v)\tilde{R}^{22}_{11}(u-v)]^{N/2} \\ \alpha_2(v) &= [R^{22}_{11}(u+v)\tilde{R}^{11}_{11}(u-v)]^{N/2} \\ \alpha_3(v) &= [R^{33}_{11}(u+v)\tilde{R}^{22}_{33}(u-v)]^{N/2}. \end{aligned} \tag{20}$$

This shows that $|\mathbf{12}\rangle$ is an eigenstate of $T(v)$ with eigenvalue $\sum_i \alpha_i(v)$. Two further sets of eigenstates can be constructed by acting with the 'creation operators' $\mathcal{L}^3_1$ and $\mathcal{L}^2_3$, respectively, i.e. we consider the states

$$|\Psi_1(v_1)\rangle = \mathcal{L}^3_1(v_1)|\mathbf{12}\rangle \text{ and } |\Psi_2(w_1)\rangle = \mathcal{L}^2_3(w_1)|\mathbf{12}\rangle. \tag{21}$$

Using the relations (18), one finds that these two states are indeed eigenstates of the transfer matrix $T(v)$ with eigenvalues

$$\begin{aligned}\Lambda_1(v) &= \frac{R_{11}^{11}(v_1-v)}{R_{11}^{33}(v_1-v)}\alpha_1(v) + \frac{R_{33}^{22}(v-v_1)}{R_{11}^{22}(v-v_1)}\alpha_2(v) + \frac{R_{33}^{33}(v-v_1)}{R_{11}^{33}(v-v_1)}\alpha_3(v)\\ \Lambda_2(v) &= \frac{R_{11}^{33}(v-w_1)}{R_{11}^{22}(v-w_1)}\alpha_1(v) + \frac{R_{22}^{22}(v-w_1)}{R_{33}^{22}(v-w_1)}\alpha_2(v) + \frac{R_{33}^{33}(w_1-v)}{R_{33}^{22}(w_1-v)}\alpha_3(v).\end{aligned} \tag{22}$$

The parameters v_1 and w_1 can be found from the requirement that $\operatorname{Res}\Lambda_1(v=v_1) = \operatorname{Res}\Lambda_2(w=w_1) = 0$. We state that the general eigenvalue equation for $T(v)$ is given by the analytic function

$$\begin{aligned}\Lambda(v) &= \prod_i \frac{R_{11}^{11}(v_i-v)}{R_{11}^{33}(v_i-v)}\prod_j \frac{R_{11}^{33}(v-w_j)}{R_{11}^{22}(v-w_j)}\alpha_1(v) + \prod_i \frac{R_{33}^{22}(v-v_i)}{R_{11}^{22}(v-v_i)}\prod_j \frac{R_{22}^{22}(v-w_j)}{R_{33}^{22}(v-w_j)}\alpha_2(v)\\ &\quad + \prod_i \frac{R_{33}^{33}(v-v_i)}{R_{11}^{33}(v-v_i)}\prod_j \frac{R_{33}^{33}(w_j-v)}{R_{33}^{22}(w_j-v)}\alpha_3(v)\\ &= \{\sinh[\eta+\epsilon_1(u+v)]\epsilon_{12}\sinh(u-v)\}^{N/2}/\sinh^N\eta\prod_i\frac{\sinh[\eta+\epsilon_1(v_i-v)]}{\epsilon_{13}\sinh(v_i-v)}\\ &\quad \times\prod_j\frac{\epsilon_{13}}{\epsilon_{12}} + \{\epsilon_{12}\sinh(u+v)\sinh[\eta+\epsilon_2(u-v)]\}^{N/2}/\sinh^N\eta\\ &\quad \times\prod_i\frac{\epsilon_{32}}{\epsilon_{21}}\prod_j\frac{\sinh[\eta+\epsilon_2(v-w_j)]}{\epsilon_{32}\sinh(v-w_j)}\\ &\quad +\{\epsilon_{13}\sinh(u+v)\epsilon_{32}\sinh(u-v)\}^{N/2}/\sinh^N\eta\prod_i\frac{\sinh[\eta+\epsilon_3(v-v_i)]}{\epsilon_{13}\sinh(v-v_i)}\\ &\quad \times\prod_j\frac{\sinh[\eta+\epsilon_3(w_j-v)]}{\epsilon_{32}\sinh(w_j-v)}.\end{aligned} \tag{23}$$

In appendix D we indicate how this procedure can be generalized to the case $q \geqslant 4$.

5. Nonlinear integral equations

We rewrite the eigenvalue equation (23) (after substituting $v \to \mathrm{i}v$ for convenience)

$$\Lambda(v) = \lambda_1(v) + \lambda_2(v) + \lambda_3(v) \tag{24}$$

with

$$\begin{aligned}\lambda_1(v) &= \frac{q_1(v+\mathrm{i}\epsilon_1\eta)}{q_1(v)}\phi_-(v-\mathrm{i}\epsilon_1\eta)\phi_+(v)\\ \lambda_2(v) &= \frac{q_2(v-\mathrm{i}\epsilon_2\eta)}{q_2(v)}\phi_-(v)\phi_+(v+\mathrm{i}\epsilon_2\eta)\\ \lambda_3(v) &= \frac{q_1(v-\mathrm{i}\epsilon_3\eta)}{q_1(v)}\frac{q_2(v+\mathrm{i}\epsilon_3\eta)}{q_2(v)}\phi_-(v)\phi_+(v)\end{aligned} \tag{25}$$

and

$$\begin{aligned}\phi_\pm(v) &= \left(\frac{\sin(v\pm\mathrm{i}u)}{\sin\eta}\right)^{N/2}\\ q_1(v) &= \prod_i^{N/2}\sin(v-v_i) \qquad q_2(v) = \prod_j^{N/2}\sin(v-w_j).\end{aligned} \tag{26}$$

The parameters v_i and w_j have to be determined from the equations

$$\lambda_1(v_i) = -\lambda_3(v_i) \qquad \text{and} \qquad \lambda_2(w_j) = -\lambda_3(w_j). \tag{27}$$

This implies that the functions

$$L_1(v) = \frac{q_1(v)}{q_2(v)}\left(1 + \frac{\lambda_3(v)}{\lambda_2(v)}\right) \qquad \text{and} \qquad L_2(v) = \frac{\phi_+(v)}{q_1(v)}\frac{\Lambda(v)}{\lambda_1(v)} \tag{28}$$

are analytic and non-zero in the vincinity of the real axis—a fact that plays an important role in the analysis of the eigenvalue equation (24) (cf appendix A).

It turns out that the largest eigenvalue of the quantum transfer matrix in the limit $N \to \infty$ is described by four complex-valued functions $a_i(x)$, $i = 1, \ldots, 4$, which satisfy a set of non-linear integral equations

$$\ln a_i(x) = -\beta\psi_i(x) + \sum_{j=1}^{4}[K_{ij} * \ln A_j](x) \qquad i = 1, \ldots, 4 \tag{29}$$

where $[f * g](x) = \int f(x-y)g(y)\,\mathrm{d}y$. The nonlinearity of the system (29) is due to the condition $A_i(x) = 1 + a_i(x)$. The form of the inhomogeneities $\psi_i(x)$ and the kernels $K_{ij}(x)$ depends on the particular grading, i.e. the choice of the parameters ϵ_α in (2). In any case the kernel $K(x)$ satisfies the symmetry property

$$K_{ij}(x) = K_{ji}(-x). \tag{30}$$

The eigenvalue of the quantum transfer matrix can be expressed in terms of the A_i functions as (see appendix B)

$$\ln \Lambda(0) = \Lambda_0 + \frac{1}{2\pi}\sum_{i=1}^{4}\int_{-\pi/2}^{\pi/2}\psi_i(x)\ln A_i(x)\,\mathrm{d}x \tag{31}$$

where the temperature dependence is entirely contained in the integral over the auxiliary functions. Expanding $K(x)$ and $\psi(x)$ in Fourier series

$$\begin{aligned} K(x) &= \frac{1}{\pi}\sum_{\substack{k=-\infty \\ k \text{ even}}}^{\infty} K(k)\mathrm{e}^{\mathrm{i}kx} \\ \psi(x) &= 2\sum_{\substack{k=-\infty \\ k \text{ even}}}^{\infty} \psi(k)\mathrm{e}^{\mathrm{i}kx} \end{aligned} \tag{32}$$

we find for the $(+,+,+)$ grading

$$\begin{aligned} K(k \geqslant 0) &= \frac{1}{1+\mathrm{e}^{\eta k}+\mathrm{e}^{2\eta k}}\begin{pmatrix} 1 & -1-\mathrm{e}^{2\eta k} & -\mathrm{e}^{2\eta k} & \mathrm{e}^{2\eta k} \\ -1-\mathrm{e}^{\eta k} & 1 & -\mathrm{e}^{\eta k} & \mathrm{e}^{\eta k} \\ -\mathrm{e}^{\eta k} & -\mathrm{e}^{2\eta k} & 1 & -1 \\ \mathrm{e}^{\eta k} & \mathrm{e}^{2\eta k} & -1 & 1 \end{pmatrix} \\ K(-k) &= K^{\mathrm{T}}(k) \\ \psi(k) &= \frac{1}{1+\mathrm{e}^{\eta k}+\mathrm{e}^{2\eta k}}(\mathrm{e}^{2\eta k}, \mathrm{e}^{\eta k}, \mathrm{e}^{\eta k}+\mathrm{e}^{2\eta k}, 1) \end{aligned} \tag{33}$$

and

$$\Lambda_0 = \beta\left[\coth\eta - 4\sum_{\substack{k=0 \\ k \text{ even}}}^{\infty}\frac{\mathrm{e}^{-\eta k}}{1+\mathrm{e}^{\eta k}+\mathrm{e}^{2\eta k}}\right]. \tag{34}$$

For the $(+,+,-)$ grading the functions $K(x)$ and $\psi(x)$ can be calculated explicitly and we find

$$\begin{aligned}
&K_{11}(x) = K_{33}(x) = K_{44}(x) = K_{34}(x) = 0\\
&K_{12}(x) = \frac{1}{\pi}[f(x+\mathrm{i}\eta) - f(x)] = K_{13}(x) = -K_{14}(x)\\
&K_{22}(x) = \frac{1}{\pi}[-f(x+\mathrm{i}\eta) - f(-x+i\eta)]\\
&K_{23}(x) = \frac{1}{\pi}[-f(x+\mathrm{i}\eta) - f(-x)] = K_{24}(x)\\
&\psi_1(x) = 2[f(x) - f(x+\mathrm{i}\eta)]\psi_2(x) = 2[f(x+i\eta) + f(-x+\mathrm{i}\eta)]\\
&\psi_3(x) = 2[f(x) + f(-x+\mathrm{i}\eta)\psi_4(x) = 2[f(-x) - f(-x+\mathrm{i}\eta)]
\end{aligned} \tag{35}$$

with

$$f(x) = \frac{1}{1-\mathrm{e}^{2\mathrm{i}x}} \tag{36}$$

and

$$\Lambda_0 = \beta \coth \eta. \tag{37}$$

Lastly we want to mention that it is possible to study the thermodynamics of the quantum chains under consideration also in the presence of certain external fields such as a chemical potential μ coupling to the particle number and a magnetic field h coupling to the spin. This leads to a modification of the quantum transfer matrix by a boundary condition depending on μ and h (cf [4]). As a consequence the eigenvalue equation (24) now reads

$$\Lambda(v) = \mathrm{e}^{\beta(\mu+h)}\lambda_1(v) + \mathrm{e}^{\beta(\mu-h)}\lambda_2(v) + \lambda_3(v) \tag{38}$$

which accounts for additional constant terms C_i on the right-hand side of the integral equations (29), namely

$$C_i = [h(1,0,1,-1) + \tilde{\mu}(-1,2,1,-1)] \tag{39}$$

where $\tilde{\mu} = \mu/3$ for the $(+,+,+)$ grading and $\tilde{\mu} = \mu$ for the $(+,+,-)$ grading. In addition Λ_0 has to be substituted with $\Lambda_0 + \beta(\mu + h)$ everywhere.

6. Low-temperature behaviour

Next study the low-temperature thermodynamics of the three-state quantum Perk–Schultz model. For low temperatures ($\beta \to \infty$) and finite η (corresponding to a non-vanishing mass gap) the right-hand side of (29) is dominated by the first term, and therefore we have to first order in T

$$\ln a_i(x) = -\beta\psi_i(x). \tag{40}$$

Inserting this in (31), expanding the logarithm, and using (14) gives

$$f = e_0 - \frac{1}{\pi\beta}\int_{-\pi/2}^{\pi/2} \psi(x)\mathrm{e}^{-\beta\psi(x)}\,\mathrm{d}x \tag{41}$$

where $e_0 = -\Lambda_0/\beta$ denotes the ground state energy. For both gradings the integrals in (41) can be evaluated in a saddle point approximation, yielding

$$f = e_0 - (\pi T/2)^{1/2}\sum_{i=1}^{4} c_i(T) \tag{42}$$

where for the $(+,+,+)$ grading

$$c_i(T) = \frac{\epsilon_0^i}{(\epsilon_1^i)^{1/2}} \mathrm{e}^{-\beta\epsilon_0^i}, \qquad i = 1, \ldots, 4 \tag{43}$$

with

$$\begin{aligned} &\epsilon_0^i = \sum_{k \text{ even}} (-)^{k/2} \epsilon(k) \qquad \epsilon_1^i = - \sum_{k \text{ even}} (-)^{k/2} k^2 \epsilon(k) \\ &\epsilon(k) = \frac{2}{1 + \mathrm{e}^{\eta k} + \mathrm{e}^{2\eta k}} (\mathrm{e}^{\eta k}, \mathrm{e}^{\eta k}, \mathrm{e}^{\eta k/2} + \mathrm{e}^{3\eta k/2}, \mathrm{e}^{\eta k}) \end{aligned} \tag{44}$$

and for the $(+,+,-)$ grading

$$\begin{aligned} &c_1(T) = c_4(T) = 0 \\ &c_2(T) = \frac{1}{1 + \mathrm{e}^{-2\eta}} \mathrm{e}^{-2\beta/(1+\mathrm{e}^{-2\eta})} \left(\frac{\cosh^3 \eta}{\sinh \eta} \right)^{1/2} \\ &c_3(T) = \frac{1}{1 + \mathrm{e}^{-\eta}} \mathrm{e}^{-2\beta/(1+\mathrm{e}^{-\eta})} \left(\frac{\cosh^3 \eta/2}{\sinh \eta/2} \right)^{1/2} . \end{aligned} \tag{45}$$

Finally we want to comment on the isotropic limit ($\eta \to 0$). Here the relevant equations have a different mathematical structure to those in the anisotropic case reflecting the different physical behaviour of the model which becomes critical in the isotropic limit. We find (cf appendix C) for the $(+,+,+)$ grading (Uimin–Sutherland model)

$$f = \left(-1 + 2 \int_0^\infty \frac{\mathrm{e}^{-k}}{1 + \mathrm{e}^k + \mathrm{e}^{2k}} \, \mathrm{d}k \right) - \frac{T^2}{2}. \tag{46}$$

This is of the form predicted by conformal field theory [16, 17]

$$f = e_0 - \frac{\pi c}{6v} T^2 \tag{47}$$

where $v = 2\pi/3$ is the sound velocity and $c = 2$ the central charge of the underlying field theory [18]. Numerical results for the free energy and specific heat at arbitrary temperatures are shown in figure 2. Note the linear temperature dependence at low temperatures in accordance with the analytic results. Furthermore, the specific heat shows a maximum and a shoulder at intermediate temperatures. This structure is due to the elementary excitations of the system. There are two such excitations [13, 14] with the same velocity, but different 'bandwidths', hence different characteristic temperatures.

For the $(+,+,-)$ grading (t–J model) we find (cf [19])

$$f = -1 - \beta^{-3/2} \frac{1}{\pi} \int_{-\infty}^{\infty} \ln(1 + \mathrm{e}^{-x^2}) \, \mathrm{d}x. \tag{48}$$

The fact that no terms of $\mathcal{O}(1/\beta)$ turn up in this expansion can be explained by the fact that the chemical potential in the t–J Hamiltonian we investigate here is equal to the lower edge of the band. Therefore the ground state is the vacuum, and the elementary excitations obey a dispersion law $\epsilon(k) \sim k^2$, which accounts for the $\beta^{-3/2}$ dependence of the free energy for low temperatures. A more detailed study of the t–J chain at arbitrary temperatures will be published in [19].

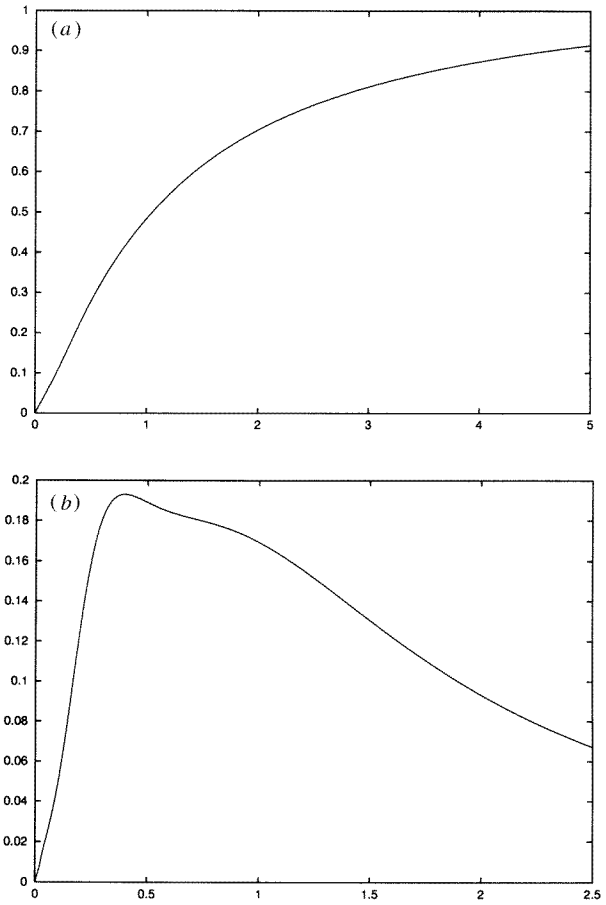

Figure 2. (*a*) Free energy of the isotropic Uimin–Sutherland model. (*b*) Specific heat of the isotropic Uimin–Sutherland model

7. Conclusion

We have studied the thermodynamics of the quantum Perk–Schultz model. We have introduced an appropriately defined quantum transfer matrix. By algebraic Bethe ansatz methods we have derived the corresponding eigenvalue equation. For the largest eigenvalue of the three-component system we have transformed these equations into a set of nonlinear integral equations. We have studied the low-temperature behaviour of the free energy of both the t–J and the Uimin–Sutherland model analytically. For the latter model we have presented numerical results for the free energy and the specific heat at arbitrary temperatures.

There are several questions related to the material presented in this paper which are currently under investigation. First we want to calculate the free energy of the t–J model for arbitary temperatures. We are also interested in studying the excited states of the models under consideration in order to obtain information about the correlation lengths at finite temperatures. Finally we want to take a closer look at the structure of the algebraic Bethe ansatz for $q \geqslant 4$ and derive information about the thermodynamics of the model in this general case.

Appendix A

Here we give a detailed account of the derivation of the system of integral equations (29). We define the following auxiliary functions:

$$\begin{aligned} a_1(v) &= \frac{\lambda_3(v)}{\lambda_2(v)} \qquad a_2(v) = [a_1(v)]^{-1} \\ a_3(v) &= \frac{\lambda_1(v)}{\lambda_2(v)+\lambda_3(v)} \qquad a_4(v) = [a_3(v)]^{-1} \end{aligned} \tag{A.1}$$

and related functions

$$A_i(v) = 1 + a_i(v). \tag{A.2}$$

Obviously, a_1 and a_2 (a_3 and a_4) are reciprocal. However, these functions will be used in different regimes of the complex plane, which are separated by singularities, namely the numbers v_i and w_j. These numbers form curved lines in the complex plane, which come close to the real axis near the origin. The functions A_1 and A_3 (A_2 and A_4) will be used above (below) these lines. In terms of these auxiliary functions we can write the functions (28) as follows:

$$\begin{aligned} L_1(v) &= \frac{q_1(v)}{q_2(v)} A_1(v) = \frac{q_1(v)}{q_2(v)} \frac{A_2(v)}{a_2(v)} \\ L_2(v) &= \frac{\phi_+(v)}{q_1(v)} \frac{A_3(v)}{a_3(v)} = \frac{\phi_+(v)}{q_1(v)} A_4(v). \end{aligned} \tag{A.3}$$

As mentioned in the text these functions are analytic and non-zero in the vicinity of the real axis. For each of the two functions we have given one representation which is valid in the upper analyticity region and one which is valid in the lower analyticity region. The equations (A.1) and (A.3) constitute six relations for the functions $a_1, \ldots, a_4, q_1, q_2$, which can be solved in terms of $A_1, \ldots, A_4$. The strategy is to Fourier transform the logarithm of these equations, eliminate the q functions between them, and obtain the transforms of $\ln a_i$ in terms of $\ln A_j$. Here we are working with (non-zero) analytic functions $f(x)$ which are π-periodic. Hence, the logarithm $\ln f(x)$ admits a Fourier series in the form

$$\ln f(x) = n\mathrm{i}x + \sum_{k=-\infty}^{\infty} f(k)\mathrm{e}^{2\mathrm{i}kx} \qquad n \in \mathbb{N}. \tag{A.4}$$

As the functions $\phi_+(v)$ and $\phi_-(v)$ have zeros close to the real axis their logarithms are described by two different sets of Fourier coefficients in the upper and lower analyticity regions:

$$\ln \phi_\pm = \begin{cases} -\dfrac{N}{2}\mathrm{i}v + \phi_\pm^+(0) + \displaystyle\sum_{k=1}^{\infty} \phi_\pm^+(k) & \mathrm{Im}(v) > 0 \\ \dfrac{N}{2}\mathrm{i}v + \phi_\pm^-(0) + \displaystyle\sum_{k=-\infty}^{0} \phi_\pm^-(k) & \mathrm{Im}(v) < 0. \end{cases} \tag{A.5}$$

with

$$\begin{aligned} \phi_\pm^+(0) &= -\frac{N}{2}\left[\ln(2\sin\eta) + \frac{\pi}{2}\mathrm{i} \pm u\right] \qquad \phi_\pm^-(0) = -\frac{N}{2}\left[\ln(2\sin\eta) - \frac{\pi}{2}\mathrm{i} \mp u\right] \\ \phi_\pm^+(k) &= -\frac{N}{2}\mathrm{e}^{\mp 2u}/k \qquad \phi_\pm^-(k) = \frac{N}{2}\mathrm{e}^{\mp 2u}/k. \end{aligned} \tag{A.6}$$

Analogously we have for the functions $q_1(v)$ and $q_2(v)$

$$\ln q_i(x) = \begin{cases} -\mathrm{i}\dfrac{N}{2}x + \displaystyle\sum_{k=0}^{\infty} q_i^+(k)\mathrm{e}^{2\mathrm{i}kx} & \mathrm{Im}(v) > 0 \\ +\mathrm{i}\dfrac{N}{2}x + \displaystyle\sum_{k=-\infty}^{0} q_i^-(k)\mathrm{e}^{2\mathrm{i}kx} & \mathrm{Im}(v) < 0. \end{cases} \tag{A.7}$$

The Fourier representation of $\ln A_i(x)$

$$\ln A_i(x) = \sum_{k=-\infty}^{\infty} A_i(k)\mathrm{e}^{2\mathrm{i}kx} \tag{A.8}$$

will be used only in the upper (lower) analyticity region for the cases $i = 1, 3$ ($i = 2, 4$). Inserting these Fourier series into the definitions (A.1) and the equations (A.3) we get for general k the following six equations:

$$\begin{aligned}
a_1(k) &= \mathrm{e}^{\eta\epsilon_3 k}q_1^{-\epsilon_3}(k) - q_1^+(k) + \mathrm{e}^{-\eta\epsilon_3 k}q_2^{\epsilon_3}(k) - \mathrm{e}^{\eta\epsilon_2 k}q_2^{-\epsilon_2}(k) + \phi_+^+(k) - \mathrm{e}^{-\eta\epsilon_2 k}\phi_+^{\epsilon_2}(k) \\
a_2(k) &= -\mathrm{e}^{\eta\epsilon_3 k}q_1^{-\epsilon_3}(k) + q_1^-(k) - \mathrm{e}^{-\eta\epsilon_3 k}q_2^{\epsilon_3}(k) + \mathrm{e}^{\eta\epsilon_2 k}q_2^{-\epsilon_2}(k) - \phi_+^-(k) + \mathrm{e}^{-\eta\epsilon_2 k}\phi_+^{\epsilon_2}(k) \\
a_3(k) &= -q_1^+(k) + \mathrm{e}^{-\eta\epsilon_1 k}q_1^{\epsilon_1}(k) - \mathrm{e}^{\eta\epsilon_2 k}q_2^{-\epsilon_2}(k) + q_2^+(k) - \phi_-^+(k) + \mathrm{e}^{\eta\epsilon_1 k}\phi_-^{-\epsilon_1}(k) \\
&\quad -\mathrm{e}^{-\eta\epsilon_2 k}\phi_+^{\epsilon_2}(k) + \phi_+^+(k) - A_1(k) \\
a_4(k) &= q_1^-(k) - \mathrm{e}^{-\eta\epsilon_1 k}q_1^{\epsilon_1}(k) + \mathrm{e}^{\eta\epsilon_2 k}q_2^{-\epsilon_2}(k) - q_2^-(k) + \phi_-^-(k) - \mathrm{e}^{\eta\epsilon_1 k}\phi_-^{-\epsilon_1}(k) \\
&\quad +\mathrm{e}^{-\eta\epsilon_2 k}\phi_+^{\epsilon_2}(k) - \phi_+^-(k) + A_2(k) - a_2(k) \\
q_1^+(k) - q_2^+(k) + A_1(k) &= q_1^-(k) - q_2^-(k) + A_2(k) - a_2(k) \\
\phi_+^-(k) - q_1^-(k) + A_4(k) &= \phi_+^+(k) - q_1^+(k) + A_3(k) - a_3(k).
\end{aligned} \tag{A.9}$$

This set of equations has to be analysed for both gradings and for $k >, <, = 0$, separately. For example for the $(+,+,+)$ grading we find for $k > 0$

$$\begin{aligned}
a_1(k) &= -q_1^+(k) + \mathrm{e}^{-\eta k}q_2^+(k) + (1 - \mathrm{e}^{-\eta k})\phi_+^+(k) \\
a_2(k) &= -\mathrm{e}^{-\eta k}q_2^+(k) + \mathrm{e}^{-\eta k}\phi_+^+(k) \\
a_3(k) &= (1 - \mathrm{e}^{-\eta k})q_1^+(k) + q_2^+(k) + (1 - \mathrm{e}^{-\eta k})\phi_+^+(k) - \phi_-^+(k) - A_1(k) \\
a_4(k) &= -\mathrm{e}^{-\eta k}q_1^+(k) + \mathrm{e}^{-\eta k}\phi_+^+(k) + A_2(k) - a_2(k) \\
q_1^+(k) - q_2^+(k) + A_1(k) &= A_2^-(k) - a_2(k) \\
A_4(k) &= \phi_+^+(k) - q_1^+(k) + A_3(k) - a_3(k)
\end{aligned} \tag{A.10}$$

for $k < 0$

$$\begin{aligned}
a_1(k) &= \mathrm{e}^{\eta k}q_1^-(k) - \mathrm{e}^{\eta k}q_2^-(k) \\
a_2(k) &= (1 - \mathrm{e}^{\eta k})q_1^-(k) + \mathrm{e}^{\eta k}q_2^-(k) - \phi_+^-(k) \\
a_3(k) &= -\mathrm{e}^{\eta k}q_1^-(k) - \phi_-^-(k) - A_1(k) \\
a_4(k) &= q_1^-(k) + (\mathrm{e}^{\eta k} - 1)q_2^-(k) - \phi_+^-(k) - \mathrm{e}^{\eta k}\phi_-^-(k) + A_2(k) - a_2(k) \\
A_1(k) &= q_1^-(k) + q_2^-(k) - A_2^-(k) - a_2(k) \\
\phi_+^-(k) - q_1^-(k) + A_4(k) &= A_3(k) - a_3(k)
\end{aligned} \tag{A.11}$$

and for $k = 0$

$$\begin{aligned}
&a_1(k=0) = q_1^-(0) - q_1^+(0) + q_2^+(0) - q_2^-(0)\\
&a_2(k=0) = q_2^-(0) - q_2^+(0) + \phi_+^+(0) - \phi_+^-(0)\\
&a_3(k=0) = q_2^+(0) - q_2^-(0) - \phi_-^+(0) + \phi_-^-(0) - A_1(k=0)\\
&a_4(k=0) = q_1^-(0) - q_1^+(0) + \phi_+^+(0) - \phi_+^-(0) + A_2(k=0) - a_2(k=0)\\
&q_1^+(0) - q_2^+(0) + A_1(k=0) = q_1^-(0) - q_2^-(0) + A_2(k=0) - a_2(k=0)\\
&\phi_+^-(0) - q_1^-(0) + A_4(k=0) = A_3(k=0) - a_3(k=0) + \phi_+^+(0) - q_1^+(0).
\end{aligned} \tag{A.12}$$

These equations can be solved for the $a_i(k)$ in terms of the $A_j(k)$, yielding

$$a_i(k) = \phi_i(k) + \sum_{j=1}^{4} K_{ij}(k) A_j(k) \tag{A.13}$$

with

$$\phi(k) = 2N\mathrm{i}\sinh(ku)\psi(k) \tag{A.14}$$

and $\psi(k)$ and $K(k)$ as in (33). After applying the inverse Fourier transform (A.13) turns into a system of integral equations for $\ln a_i(x)$ in terms of $\ln A_j(x)$. Performing the limit $N \to \infty$ in these equations yields (29). The grading $(+,+,-)$ can be treated in a similar manner.

Appendix B

In this appendix we calculate the eigenvalue of the quantum transfer matrix in terms of the auxiliary functions (A.1). From (A.1) we see that we can write $\Lambda(v)$ as

$$\Lambda(v) = \frac{q_1(v + \mathrm{i}\epsilon_1\eta)}{q_1(v)} \phi_-(v - \mathrm{i}\epsilon_1\eta)\phi_+(v)A_4(v). \tag{B.1}$$

Expanding $\ln\Lambda(v)$ in a Fourier series gives

$$\ln\Lambda(v) = \sum_{k=-\infty}^{\infty} \Lambda(k)\mathrm{e}^{2\mathrm{i}kx}. \tag{B.2}$$

Upon inserting this series in (B.1) we find

$$\Lambda(k) = \mathrm{e}^{-\epsilon_1\eta} q_1^{\epsilon_1}(k) - q_1^-(k) + \mathrm{e}^{\epsilon_1\eta}\phi_-^{-\epsilon_1}(k) + \phi_+^-(k) + A_4(k). \tag{B.3}$$

As in appendix A these equations have to be analysed for both gradings, separately. For the $(+,+,+)$ grading we have

$$\Lambda(k) = \begin{cases} \mathrm{e}^{-\eta k} q_1^+(k) + A_4(k) & k > 0\\ -q_1^-(k) + \mathrm{e}^{\eta k}\phi_-^-(k) + \phi_+^-(k) + A_4(k) & k < 0\\ q_1^+(0) - q_1^-(0) + \phi_-^-(k) + \phi_+^-(k) + A_4(k) & k = 0. \end{cases} \tag{B.4}$$

The coefficients $\phi_\alpha^\beta(k)$ are given in (A.6); the $q_i^\pm(k)$ can be calculated from (A.9). One finds

$$\Lambda(k) = \Lambda_0(k) + \sum_{i=1}^{4} \psi_i(-k)A_i(k) \tag{B.5}$$

with $\psi_i(k)$ as in (33) and

$$\Lambda_0(k) = \begin{cases} \mathrm{e}^{-\eta k}\phi_-^+(k) + N\dfrac{\sinh(ku)}{k}\dfrac{\mathrm{e}^{-\eta k}}{1+\mathrm{e}^{\eta k+2\eta k}} & k>0 \\ \mathrm{e}^{\eta k}\phi_+^-(k) + N\dfrac{\sinh(ku)}{k}\dfrac{\mathrm{e}^{3\eta k}}{1+\mathrm{e}^{\eta k+2\eta k}} & k<0 \\ \phi_-^+(0)+\phi_+^-(0)+4Nu/3 & \mathrm{k}=0. \end{cases} \tag{B.6}$$

Inserting this in (B.2) yields

$$\begin{aligned} \ln\Lambda(x) = {} & \ln\phi_-(x+\mathrm{i}\eta) + \ln\phi_+(x-\mathrm{i}\eta) + 2N\sum_{\substack{k=0\\ k\ \text{even}}}^{\infty}\frac{\sinh(ku)}{k}\frac{\mathrm{e}^{-\eta k}}{1+\mathrm{e}^{\eta k}+\mathrm{e}^{2\eta k}}\mathrm{e}^{\mathrm{i}kx} \\ & +2N\sum_{\substack{k=0\\ k\ \text{even}}}^{\infty}\frac{\sinh(ku)}{k}\frac{\mathrm{e}^{3\eta k}}{1+\mathrm{e}^{\eta k}+\mathrm{e}^{2\eta k}}\mathrm{e}^{\mathrm{i}kx} \\ & +\frac{1}{2\pi}\sum_{i=1}^{4}\int_{-\pi/2}^{\pi/2}\psi_i(y-x)\ln A_i(y)\,\mathrm{d}y. \end{aligned} \tag{B.7}$$

Taking the limit $N\to\infty$ and letting $x=0$ we find (31). The $(+,+,-)$ grading can be treated in a similar manner.

Appendix C

In this appendix we comment on the isotropic limit of the equations (29) and (31). In order to perform the limit $\eta\to 0$ we have to rescale the space and momentum coordinates according to $x\to\eta x$ and $k\to k/\eta$ and perform the substitution $f\to\eta f$. Both equations retain their structure, but now the integral kernel $K(x)$ in (29) is given by

$$K(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}K(k)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k \qquad K(k) = K^{\mathrm{T}}(-k) \tag{C.1}$$

where

$$K(k>0) = \frac{1}{1+\mathrm{e}^k+\mathrm{e}^{2k}}\begin{pmatrix} 1 & -\mathrm{e}^k-\mathrm{e}^{2k} & -\mathrm{e}^{2k} & \mathrm{e}^{2k} \\ -1-\mathrm{e}^k & 1 & -\mathrm{e}^k & \mathrm{e}^k \\ -\mathrm{e}^k & -\mathrm{e}^{2k} & 1 & -1 \\ \mathrm{e}^k & \mathrm{e}^{2k} & -1 & 1 \end{pmatrix} \tag{C.2}$$

for the $(+,+,+)$ grading and

$$K(k>0) = \begin{pmatrix} 0 & -1+\mathrm{e}^{-k} & -1+\mathrm{e}^{-k} & 1-\mathrm{e}^{-k} \\ 0 & -\mathrm{e}^{-k} & -\mathrm{e}^{-k} & \mathrm{e}^{-k} \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{C.3}$$

for the $(+,+,-)$ grading. For the quantities Λ_0 and $\psi(x)$ in (31) we find

$$\begin{aligned} \Lambda_0 &= \beta\left[1-2\int_0^{\infty}\frac{\mathrm{e}^{-k}}{1+\mathrm{e}^k+\mathrm{e}^{2k}}\,\mathrm{d}k\right] \\ \psi(x) &= \int_{-\infty}^{\infty}\frac{(\mathrm{e}^{2k},\mathrm{e}^k,\mathrm{e}^k+\mathrm{e}^{2k},1)}{1+\mathrm{e}^k+\mathrm{e}^{2k}}\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k \end{aligned} \tag{C.4}$$

for the $(+,+,+)$ grading and

$$\Lambda_0 = \beta$$
$$\psi(x) = \left(-\frac{1}{x(x+\mathrm{i})}, \frac{2}{x^2+1}, \frac{1}{x(x-\mathrm{i})}, -\frac{1}{x(x-\mathrm{i})}\right) \tag{C.5}$$

for the $(+,+,-)$ grading.

The low-temperature behaviour of the free energy in the $(+,+,+)$ case can be deduced from the observation that the positions of the maxima of the integrands of the integral in (31) scale like $\pm\ln\beta$. We are therefore interested in the asymptotic behaviour of the function $\psi(x)$. This can be determined by taking the limit $x \to \pm\infty$ in (C.4), and noting that the asymptotic behaviour is determined by the poles at $\pm 2\pi\mathrm{i}/3$, respectively

$$\psi(x \to \pm\infty) = \frac{2\pi}{\sqrt{3}}(\mathrm{e}^{\pm 2\pi\mathrm{i}/3}, 1, 1+\mathrm{e}^{\pm 2\pi\mathrm{i}/3}, \mathrm{e}^{\pm 2\pi\mathrm{i}/3})\mathrm{e}^{\mp 2\pi x/3}. \tag{C.6}$$

This implies that for $x = \mathcal{O}(\ln\beta)$ we have $\psi'(x) = (\mp 2\pi/3)\psi(x)$, and we can write the integral in (31) as

$$\frac{1}{2\pi}\sum_{i=1}^{4}\int_{-\infty}^{\infty}\psi_i(x)\ln A_i(x)\,\mathrm{d}x$$
$$= \frac{3}{4\pi^2}\sum_{i=1}^{4}\left[\int_{-\infty}^{0}\psi_i'(x)(x)\ln A_i(x)\,\mathrm{d}x - \int_{0}^{\infty}\psi_i'(x)(x)\ln A_i(x)\,\mathrm{d}x\right]. \tag{C.7}$$

These integrals can be further transformed by integrating by parts, inserting the integral equation (40), and using the fact that the contributions of the kernel K cancel due to the symmetry (30). We obtain

$$\sum_{i=1}^{4}\int_{0}^{\infty}\psi_i'(x)\ln A_i(x)\,\mathrm{d}x = \frac{1}{2\beta}\sum_{i=1}^{4}\int_{0}^{\infty}[-\beta\psi_i'(x)\ln A_i(x) + \beta\psi_i'(x)\ln A_i(x)]\,\mathrm{d}x$$
$$= \frac{1}{2\beta}\sum_{i=1}^{4}\int_{0}^{\infty}[\ln a_i'(x)\ln A_i(x) + \ln a_i(x)\ln A_i'(x)]\,\mathrm{d}x$$
$$= \frac{1}{2\beta}\sum_{i=1}^{4}\int_{a_i(0)}^{a_i(\infty)}\left[\frac{\ln(1+a_i)}{a_i} - \frac{\ln a_i}{1+a_i}\right]\mathrm{d}a_i = \frac{1}{\beta}\sum_{i=1}^{4}L_+(a_i(\infty)) = \frac{\pi^2}{3\beta}. \tag{C.8}$$

We have introduced the dilogarithmic function

$$L_+(z) = \frac{1}{2}\int_0^z\left[\frac{\ln(1+y)}{y} - \frac{\ln y}{1+y}\right]\mathrm{d}y \tag{C.9}$$

which satisfies the functional equation

$$L_+(z) + L_+(1/z) = \pi^2/6. \tag{C.10}$$

A similar calculation can be performed for the second integral in (C.7), and we eventually find (46).

Appendix D

Here we want to generalize the eigenvalue equation for the quantum transfer matrix of the Perk–Schultz model derived in section 4 for $q = 3$ to the general case. For general q the action of the monodromy matrix on the reference state $|\mathbf{12}\rangle$ is determined by

$$\begin{aligned} &\mathcal{L}_i^i(v)|\mathbf{12}\rangle = \alpha_i(v)|\mathbf{12}\rangle \qquad i = 1, \ldots, q \\ &\mathcal{L}_i^j(v)|\mathbf{12}\rangle = 0 \qquad i \neq j, i \neq 1, j \neq 2 \end{aligned} \tag{D.1}$$

where $\alpha_1(v)$ and $\alpha_2(v)$ are as in (20) and

$$\alpha_i(v) = [R_{11}^{ii}(v+u)\tilde{R}_{ii}^{22}(v-u)]^{N/2} \qquad i \geqslant 3. \tag{D.2}$$

(D.1) implies that the operators $\mathcal{L}_1^j$ and $\mathcal{L}_j^2$ with $3 \leqslant j \leqslant q$ can be considered as creation operators. Using the commutation relations (18) it is straightforward to establish that the states

$$|\psi_1^j(v_1^{(j)})\rangle = \mathcal{L}_1^j(v_1^{(j)})|\mathbf{12}\rangle \qquad \text{and} \qquad |\psi_j^2(w_1^{(j)})\rangle = \mathcal{L}_j^2(w_1^{(j)})|\mathbf{12}\rangle \tag{D.3}$$

are eigenstates of the transfer matrix $T(v)$ with eigenvalues

$$\begin{aligned} \Lambda_1^j(v) &= \frac{R_{11}^{11}(v_1^{(j)} - v)}{R_{11}^{jj}(v_1^{(j)} - v)}\alpha_1(v) + \frac{R_{jj}^{22}(v - v_1^{(j)})}{R_{11}^{22}(v - v_1^{(j)})}\alpha_2(v) + \sum_{i=3}^{q}\frac{R_{jj}^{ii}(v - v_1^{(j)})}{R_{11}^{ii}(v - v_1^{(j)})}\alpha_i(v) \\ \Lambda_j^2(v) &= \frac{R_{11}^{jj}(v - w_1^{(j)})}{R_{11}^{22}(v - w_1^{(j)})}\alpha_1(v) + \frac{R_{22}^{22}(v - w_1^{(j)})}{R_{jj}^{22}(v - w_1^{(j)})}\alpha_2(v) + \sum_{i=3}^{q}\frac{R_{ii}^{jj}(w_1^{(j)} - v)}{R_{ii}^{22}(w_1^{(j)} - v)}\alpha_3(v). \end{aligned} \tag{D.4}$$

This implies that the general eigenvalue is of the form

$$\begin{aligned} \Lambda(v) = &\left[\prod_{i=3}^{q}\prod_{k}\frac{R_{11}^{11}(v_k^{(i)} - v)}{R_{11}^{ii}(v_k^{(i)} - v)}\prod_{j=3}^{q}\prod_{l}\frac{R_{11}^{ii}(v - w_l^{(j)})}{R_{11}^{22}(v - w_l^{(j)})}\right]\alpha_1(v) \\ &\times\left[\prod_{i=3}^{q}\prod_{k}\frac{R_{jj}^{22}(v - v_k^{(i)})}{R_{11}^{22}(v - v_k^{(i)})}\prod_{j=3}^{q}\prod_{l}\frac{R_{22}^{22}(v - w_l^{(j)})}{R_{ii}^{22}(v - w_l^{(j)})}\right]\alpha_2(v) \\ &+\sum_{n=3}^{q}\left[\prod_{i=3}^{q}\prod_{k}\frac{R_{jj}^{nn}(v - v_k^{(i)})}{R_{11}^{nn}(v - v_k^{(i)})}\prod_{j=3}^{q}\prod_{l}\frac{R_{nn}^{jj}(w_l^{(j)} - v)}{R_{nn}^{22}(w_l^{(j)} - v)}\right]\alpha_3(v). \end{aligned} \tag{D.5}$$

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